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More on superstrings in $AdS_3 \times \mathcal{N}$

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Abstract: We study superstring theories on $AdS_3 \times \mathcal{N}$ backgrounds yielding $N = 2, 3, 4$ extended superconformal symmetries in the dual boundary CFT. In each case the necessary constraints on the internal worldsheet theory $\mathcal{N}$ are found.

Keywords: Superstring Vacua, Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

The special status of the AdS_3/CFT_2 duality in the AdS/CFT context stems from the fact that the Virasoro generators of the boundary Conformal Field Theory (CFT) can be built exactly from operators in string theory on an $\mathrm{SL}(2,\mathbb{R}) \times \mathcal{N}$ worldsheet CFT. This building was done in [1, 2], where it was also shown that an affine algebra in $\mathcal{N}$ can be uplifted to a similar affine algebra in the boundary, with a different level. These constructions are particularly interesting in the supersymmetric case, where an adequate field content in $\mathcal{N}$ allows to enlarge the boundary Virasoro symmetry to Superconformal Field Theory (SCFT) algebras with various numbers of supersymmetries. This was also shown in [1], where the model $\mathrm{SL}(2,\mathbb{R})_k \times \mathrm{SU}(2)_k \times T^4$ was used to construct a small $N = 4$ SCFT algebra in the boundary theory[1] (see [3] for a different approach).

Following that work, several backgrounds were explored yielding other amounts of boundary supersymmetry. The extended ($N > 1$) SCFT algebras in two dimension have been classified in [4, 5], and are $N = 2, 3$ and two types of $N = 4$, small and large. Their R-symmetries are affine U(1), SU(2), SU(2) and $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, respectively, with levels depending on the Virasoro central charge. In the $AdS_3 \times \mathcal{N}$ backgrounds studied, the theory $\mathcal{N}$ contains these affine R-symmetries, which are then uplifted to the boundary.

[1]For simplicity, we discuss only the holomorphic sector of the worldsheet and the two dimensional spacetime CFT. The full theory also has a right-handed algebra whose structure depends on the type of string theory considered: IIA, IIB, heterotic, etc.

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AdS_3 backgrounds yielding $N = 2$ spacetime SCFT were studied in [6, 7]. For the N=3 case, three different backgrounds were proposed: $\mathcal{N}$= SU(3)/ U(1) and $\mathcal{N}$= SO(5)/ SO(3) in [8], and $\mathcal{N}$=(SU(2) × SU(2) × U(1))/Z_2 in [9]. Small $N = 4$ was mentioned above. Finally, for large $N = 4$ SCFT, a model was studied in [10] with $\mathcal{N}$ = SU(2) × SU(2) × U(1), which is the minimal field content required for the boundary R-symmetries.

All these constructions provide only *sufficient* conditions to yield boundary extended SCFTs. The purpose of this work is to study the *necessary* constraints imposed on $\mathcal{N}$ by the existence of each of the extended SCFT algebras in the boundary theory.

Our results strengthen the relationship between AdS_3 boundary supersymmetry and worldsheet symmetries of the internal CFT $\mathcal{N}$. This is similar to the case of string compactified to Minkowski space, where long established results show the intimate connection between spacetime supersymmetry and the symmetries of the compact sector [11, 12, 13, 14, 15, 16]. In this work we will use CFT techniques similar to those in [11, 12]. This makes the results hold for general CFT backgrounds, not necessarily having a geometrical picture as target spaces of non-linear σ-models.

Superstring theory on $AdS_3 \times \mathcal{N}$ afford boundary SCFT algebras in the NS sector. This emerges naturally when supercharges are built from spin fields creating the Ramond sector of the worldsheet CFT, as for flat space superstrings [17], and is in accord with expectations from AdS_3 supergravity analysis [18].

For $N = 2$ and small and large $N = 4$, we will show that the sufficient conditions stated in [6, 1, 10], respectively, are also necessary. For $N = 3$ we find a set of necessary conditions, which are an SU(2)-covariant version of a sufficient condition given in [8] to enlarge the $N = 2$ SCFT, obtained from backgrounds of the type [6, 7], to $N = 3$.

The plan of the work is as follows. Sections 2, 3 and 4 deal with the $N = 2$, $N = 3$ and $N = 4$ cases, respectively. In section 5 we present a short discussion. Appendix A is devoted to proving some properties of the R-currents. In appendix B, as an illustration, we show how the results are realized explicitly for $N = 3$ in the background $\mathcal{N} = (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1))/Z_2$.

2. Spacetime $N = 2$ supersymmetry

In this section we will show that any critical superstring vacuum of the form $AdS_3 \times \mathcal{N}$ in which the spacetime theory has (at least) $N = 2$ superconformal supersymmetry must be of the form proposed in [6, 7]. Namely, the internal CFT $\mathcal{N}$ contains an affine U(1) symmetry, and in the CFT quotient $\mathcal{N}$/ U(1) the worldsheet superconformal symmetry is extended to $N = 2$ supersymmetry.

Let us consider the global part of the $N = 2$ spacetime superalgebra in the NS sector:

$$\{Q_r^+, Q_s^-\} = 2L_{r+s} + (r - s)R_0\,, \tag{2.1}$$

$$[L_m, L_n] = (m - n)L_{m+n}\,, \tag{2.2}$$

$$[L_m, Q_r^\pm] = \left(\frac{m}{2} - r\right) Q_{m+r}^\pm\,, \tag{2.3}$$

$$[R_0, Q_r^\pm] = \pm Q_r^\pm\,, \tag{2.4}$$

with $r, s = \pm\frac{1}{2}$, $m, n = 0, \pm 1$, and all other (anti)commutators vanishing.

The above spacetime operators are given by contour integrals of dimension-1 local operators on the worldsheet. We assume that the global part $L_{0,\pm1}$ of the Virasoro algebra in spacetime along with the higher modes L_n are given by the construction presented in [1], so that, up to picture-changing [17],

$$\begin{aligned} L_0 &= -\oint J^3 = -\oint e^{-\phi}\psi^3\,, \\ L_{\pm1} &= -\oint (J^1 \pm iJ^2) = -\oint e^{-\phi}(\psi^1 \pm i\psi^2)\,, \end{aligned} \tag{2.5}$$

The field ϕ is the bosonized superghost and the superfields $\psi^A + \theta J^A$, $A = 1,2,3$, are the affine currents of a supersymmetric SL$(2,\mathbb{R})$ WZW model at level k. Their OPEs are:[2]

$$\begin{aligned} J^A(z)J^B(w) &\sim \frac{\frac{k}{2}\eta^{AB}}{(z-w)^2} + \frac{i\epsilon^{ABC}\eta_{CD}J^D(w)}{z-w}\,, \\ J^A(z)\psi^B(w) &\sim \frac{i\epsilon^{ABC}\eta_{CD}\psi^D(w)}{z-w}\,, \\ \psi^A(z)\psi^B(w) &\sim \frac{\frac{k}{2}\eta^{AB}}{z-w}\,, \end{aligned} \tag{2.6}$$

where $\eta^{AB} = (++-)$ and $\epsilon^{123} = 1$. As usual in supersymmetric WZW models, we can define the currents

$$j^A = J^A + \frac{i}{k}\epsilon^{ABC}\eta_{BD}\eta_{CE}\psi^D\psi^E\,, \tag{2.7}$$

which form an SL$(2,\mathbb{R})$ affine algebra at level $k+2$ and have regular OPE with the free fermions ψ^A. The central charge of the AdS_3 sector is $c = 9/2 + 6/k$, so for a critical theory, $\mathcal{N}$ must have $c_{\mathcal{N}} = 21/2 - 6/k$.

R_0 is the zero mode of the U(1) R-current of the $N = 2$ algebra in spacetime. The higher modes R_n can be obtained, say, from the commutators of R_0 with L_n, once the latter are introduced. Alternatively, they can be obtained by the procedure described in [1] to uplift an affine current from the worldsheet to the boundary spacetime theory (see below).

Let $\psi^0 + \theta J^0$ be the worldsheet supercurrent corresponding to R_0; we have, up to picture-changing,

$$R_0 = \sqrt{2k}\oint J^0 = \sqrt{2k}\oint e^{-\phi}\psi^0\,, \tag{2.8}$$

where J^0 and ψ^0 are orthogonal and canonically normalized (see appendix A):

$$\begin{aligned} J^0(z)J^0(w) &\sim \frac{1}{(z-w)^2}\,, \\ \psi^0(z)\psi^0(w) &\sim \frac{1}{z-w}\,, \\ J^0(z)\psi^0(w) &\sim 0\,. \end{aligned} \tag{2.9}$$

[2]Indices that go from 1 to 3 will be indicated with capital letters $(A, B \ldots)$. For those going from 0 to 3 later we will use lower case letters $(i, j \ldots)$.

To see that the choice (2.9) leads to the normalization of (2.8), recall [1] that the higher modes of the R-current have the form

$$R_n = a \oint J^0 \gamma^n \,, \tag{2.10}$$

and they satisfy

$$[R_m, R_n] = a^2 p m \delta_{n+m,0} \,, \tag{2.11}$$

with

$$p = \oint \frac{\partial_z \gamma}{\gamma} \,, \tag{2.12}$$

where γ is the zero-dimension field of the (β, γ) pair appearing in the Wakimoto free-field representation of the algebra (2.6). We want to show that $a = \sqrt{2k}$. Indeed, consistency of the $N = 2$ algebra [4, 19] in spacetime implies $a^2 p = c_{st}/3$, where c_{st} is the central charge of the Virasoro algebra in the dual boundary CFT, given by $c_{st} = 6kp$.[3] This fixes $a = \pm\sqrt{2k}$.[4]

As for the relation between this U(1) current and the three $SL(2,\mathbb{R})$ currents, in appendix A we show that the commutation relations in spacetime (2.1)–(2.4) and the fact that the worldsheet theory is supersymmetric force $\psi^0 + \theta J^0$ to lie *entirely* in the internal CFT $\mathcal{N}$.

2.1 Properties of the spacetime supercharges

Regarding the four spacetime supercharges $Q_r^{\pm}$, we will only assume that, as for superstrings in flat space [17], they are obtained from operators that create the worldsheet Ramond sector:

$$Q_r^{\pm} = b_r^{\pm} \oint e^{-\frac{\phi}{2}} S_r^{\pm} \,, \qquad r = \pm\frac{1}{2} \,, \tag{2.13}$$

where $S_r^{\pm}$ are spin fields [20, 17] and $b_r^{\pm}$ are constants. Since unbroken worldsheet supersymmetry requires $G_0^2 = L_0^{ws} - c/24 = 0$ on the Ramond ground state, we have $\Delta(S_r^{\pm}) = c/24 = 5/8$. This is compatible with the on-shell condition $\Delta(Q) = 0$ for the Q's in (2.13). In the presence of spin fields, the fermionic parts of the superfields become double-valued, i.e., integer modded on the plane, and the bosonic fields remain single-valued.

In our case, we have identified four free fermions ψ^i, $i = 0, 1, 2, 3$, which are the lower components of superfields. Since, at the moment, we are not given further data on the field content of the worldsheet theory $\mathcal{N}$, we only know that the whole set of spin fields is in a representation of the algebra satisfied by the zero modes of ψ^i. This is the four dimensional Clifford algebra:

$$\{\psi_0^i, \psi_0^j\} = g^{ij} k^i \,, \tag{2.14}$$

with $g^{ij} = (+,+,+,-)$, $k^0 = 1$ and $k^A = k/2$. In particular, we can decompose the whole set of spin fields into irreducible representations (irreps) of (2.14). By a result proved

[3]The reader is referred to [1] for details of the construction.

[4]In the following we will consider only the expressions for $a = +\sqrt{2k}$, but everything holds for the other choice, changing signs appropriately.

in [21], all the irreps of (2.14) have dimension 4 and are all equivalent. Hence the OPE of the spin fields with the four fermions is [20, 22]:

$$\psi^i(z) S_{\varepsilon_1,\varepsilon_2,\lambda}(w) \sim \frac{(\psi_0^i)_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1',\varepsilon_2'} S_{\varepsilon_1',\varepsilon_2',\lambda}(z)}{(z-w)^{1/2}}\,, \tag{2.15}$$

where λ is an index indicating to which particular irrep we refer, and $\varepsilon_{1,2} = \pm 1/2$. The spacetime algebra (2.1)–(2.4) implies that $S_r^\pm$ in (2.13) form such an irrep (this will be described explicitely below).

By Wick rotating ψ^3, these irreps realize (anti)spinorial representations of the level-2 $\mathrm{SO}(4)_2$ affine algebra constructed out of bilinears of $\psi^i(z)$. Since the $\mathrm{SO}(4)_2$ currents are bosonic dimension-1 fields, their OPEs with the spin fields are single-valued. For such (anti)spinorial representations of $\mathrm{SO}(4)_2$, the weights are $\pm 1/2$. Choosing for the Cartan subalgebra the fields

$$\partial H_1 = \frac{2}{k}\psi^1\psi^2\,, \qquad \partial H_2 = -i\sqrt{\frac{2}{k}}\psi^0\psi^3\,, \tag{2.16}$$

with

$$H_I(z) H_J(w) \sim -\delta_{IJ}\log(z-w)\,, \qquad I, J = 1, 2\,, \tag{2.17}$$

and $H_1^\dagger = H_1$, $H_2^\dagger = -H_2$, we must have

$$i\partial H_{1,2}(z) S_{\varepsilon_1,\varepsilon_2,\lambda}(w) \sim \varepsilon_{1,2}\frac{S_{\varepsilon_1,\varepsilon_2,\lambda}(w)}{z-w}\,, \qquad \varepsilon_{1,2} = \pm\frac{1}{2}\,. \tag{2.18}$$

It follows that the fields $S_{\varepsilon_1,\varepsilon_2,\lambda}(z)$ can be written as

$$S_{\varepsilon_1,\varepsilon_2,\lambda} = e^{i\varepsilon_1 H_1 + i\varepsilon_2 H_2} e^{i\pi\varepsilon_2 N_1}\,\tilde{\Sigma}_\lambda\,, \tag{2.19}$$

where we have written explicitly the cocycle $e^{i\pi\varepsilon_2 N_1}$ [23], which is necessary for (2.19) to yield a good representation of (2.15). The number operator N_1 is given by

$$N_1 = i\oint \partial H_1(z)\,. \tag{2.20}$$

Equations (2.18) -(2.19) imply that

$$H_I(z)\tilde{\Sigma}_\lambda(w) \sim 0\,, \tag{2.21}$$

and from this we have in turn[5]

$$\psi^i(z)\tilde{\Sigma}_\lambda(w) \sim 0\,. \tag{2.22}$$

For future reference we remind that

$$e^{\pm iH_1} = \frac{1}{\sqrt{k}}(\psi^1 \mp i\psi^2)\,, \tag{2.23}$$

$$e^{\pm i\pi N_1} e^{\pm iH_2} = \frac{1}{\sqrt{2}}\psi^0 \mp \frac{1}{\sqrt{k}}\psi^3\,. \tag{2.24}$$

[5]For example, $\psi^0(z)\tilde{\Sigma}_\lambda(w) \sim \frac{(e^{iH_2(z)} - e^{-iH_2(z)})}{i\sqrt{2}}\tilde{\Sigma}_\lambda(w) \sim 0$, and so on.

Note that in (2.19) we have obtained for $S_{\varepsilon_1,\varepsilon_2,\lambda}$ the structure usually assumed, with $\tilde{\Sigma}_\lambda$ generally being a spin field for the other fermionic fields present in the theory. But the path we have taken is meant to stress that relations (2.19), (2.21) and (2.22) hold under the general property of $S_{\varepsilon_1,\varepsilon_2,\lambda}$ being a representation of the algebra of the four fermions ψ^i, regardless of the further structure of the $\mathcal{N}$ CFT.

According to (2.3), the charge of $S_r^\pm$ ($r = \pm\frac{1}{2}$) under L_0 is $-r$. Since $e^{i\varepsilon_1 H_1}$ and $e^{i\varepsilon_2 H_2}$ have charges ε_1 and 0 under L_0, respectively, we conclude that the identification $r = -\varepsilon_1$ should be made, and that $\tilde{\Sigma}_\lambda$ is uncharged under L_0. This is consistent with the action of L_{+1} (L_{-1}), which lowers (raises) the eigenvalue of L_0 by one, provided that $\tilde{\Sigma}_\lambda$ is untouched by $J^\pm$. It follows that[6]

$$J^A(z)\tilde{\Sigma}_\lambda(w) \sim 0\,. \tag{2.25}$$

Having identified the L_0-charge of $S_{\varepsilon_1,\varepsilon_2,\lambda}$ as ε_1, we expect $S_r^\pm$ to be obtained from $S_{-r,\varepsilon_2,\lambda}$. Imposing that the supercharges $Q_r^\pm$ of (2.13) satisfy (2.1), along with $\{Q_r^\pm, Q_s^\pm\} = 0$, fixes[7]

$$S_r^\pm = e^{-ir(H_1 \mp H_2)}\tilde{\Sigma}^\pm\,, \qquad r = \pm\frac{1}{2}\,, \tag{2.26}$$

where we have relabelled $\tilde{\Sigma}_\lambda$ accordingly. Consistency of the algebra (2.1)–(2.4) requires

$$\begin{aligned} \tilde{\Sigma}^+(z)\tilde{\Sigma}^-(w) &\sim \frac{1}{(z-w)^{3/4}}\,, \\ \tilde{\Sigma}^\pm(z)\tilde{\Sigma}^\pm(w) &\sim O(w)(z-w)^{3/4}\,. \end{aligned} \tag{2.27}$$

Finally, the constants $b_r^\pm$ in (2.13) are determined to be $(4k)^{1/4}$ up to phases.

According to (2.4) and (2.8), the operators $S_r^\pm$ are charged under J^0 with charges $\pm 1/\sqrt{2k}$. But from the results of appendix A, we know that the operators $e^{-ir(H_1 \mp iH_2)}$ are neutral under J^0, so the charges are in $\tilde{\Sigma}_r^\pm$. Writing

$$J_0 = i\partial Y\,, \tag{2.28}$$

with

$$Y(z)Y(w) \sim -\log(z-w)\,, \tag{2.29}$$

we must have

$$\tilde{\Sigma}^\pm = e^{\pm i\sqrt{\frac{1}{2k}}Y}\Sigma^\pm\,, \tag{2.30}$$

and

$$Y(z)\Sigma^\pm(w) \sim 0\,. \tag{2.31}$$

Collecting our results until now, we have

$$S_r^\pm = e^{-ir(H_1 \mp H_2)}e^{\pm i\sqrt{\frac{1}{2k}}Y}\Sigma^\pm\,, \tag{2.32}$$

[6] An L_0 charge of $-r$ for $S_{\varepsilon_1,\varepsilon_2,\lambda}$ could also have been obtained by choosing $\varepsilon_1 = r$ and letting $\tilde{\Sigma}_\lambda$ carry charge $-2r$ under L_0, but this option is inconsistent with the action of $L_{\pm1}$.

[7] We omit the cocycles from now on.

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and from (2.22), (2.25) and (2.31) it follows that $\Sigma^{\pm}$ belong entirely to $\mathcal{N}/\,\mathrm{U}(1)$, i.e.,

$$J^i(z)\Sigma^{\pm}(w) \sim \psi^i(z)\Sigma^{\pm}(w) \sim 0\,. \tag{2.33}$$

Moreover,

$$\Delta(\Sigma^{\pm}) \equiv \frac{\beta^2}{2} = \frac{3}{8} - \frac{1}{4k} = \frac{c_{\mathcal{N}/\,\mathrm{U}(1)}}{24}\,, \tag{2.34}$$

as should be for operators that create the Ramond sector ground state of $\mathcal{N}/\,\mathrm{U}(1)$ with unbroken supersymmetry.

After the introduction of the $e^{\pm i\sqrt{\frac{1}{2k}}Y}$ factors in $\tilde{\Sigma}^{\pm}$, eq. (2.27) turns into

$$\begin{aligned}\Sigma^{-}(z)\Sigma^{+}(w) &\sim \frac{1}{(z-w)^{\beta^2}}\,,\\ \Sigma^{\pm}(z)\Sigma^{\pm}(w) &\sim (z-w)^{\beta^2}O^{\pm}(w)\,,\end{aligned} \tag{2.35}$$

where $\Delta(O^{\pm}) = 3/2 - 1/k$. We can now use standard techniques [11, 24]. Consider the four-point function

$$f(z_j) = \langle \Sigma^{-}(z_1)\Sigma^{+}(z_2)\Sigma^{-}(z_3)\Sigma^{+}(z_4)\rangle\,. \tag{2.36}$$

Using (2.34) and $\mathrm{SL}(2,\mathbb{C})$ invariance, it can be written as

$$f(z_j) = \left(\frac{z_{13}z_{24}}{z_{12}z_{34}z_{14}z_{23}}\right)^{\beta^2}\tilde{f}(x)\,, \tag{2.37}$$

where $z_{jk} = z_j - z_k$ and $x \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}$. As the points z_i coincide pairwise, the OPEs (2.35) imply that $\tilde{f}$ is an analytic function and is bounded for $x \to \infty$, hence it is a constant. Expanding as $z_{12} \to 0$, (2.37) becomes

$$f(z_j) = z_{12}^{-\beta^2}z_{34}^{-\beta^2}\left(1 + \beta^2\frac{z_{12}z_{43}}{z_{23}z_{24}}\right)\,, \tag{2.38}$$

where the first term and (2.35) fix $\tilde{f}$ to 1. The presence of a second term in (2.38) implies that in the $\Sigma^{-}\Sigma^{+}$ OPE expansion there is a dimension-1 field M,

$$\Sigma^{-}(z_1)\Sigma^{+}(z_2) \sim \frac{1}{z_{12}^{\beta^2}}\left[1 + z_{12}\frac{M(z_2)}{2}\right]\,, \tag{2.39}$$

whose three-point function with $\Sigma^{-}\Sigma^{+}$ is determined from (2.38) to be

$$\langle M(z_2)\Sigma^{-}(z_3)\Sigma^{+}(z_4)\rangle = 2\beta^2 z_{34}^{1-\beta^2}z_{23}^{-1}z_{24}^{-1}\,. \tag{2.40}$$

Taking the limits $z_{34}, z_{23}, z_{24} \to 0$, we obtain

$$M(z)M(w) \sim \frac{4\beta^2}{(z-w)^2}\,, \tag{2.41}$$

and

$$M(z)\Sigma^{\pm}(w) \sim \pm\frac{2\beta^2\Sigma^{\pm}}{z-w}\,. \tag{2.42}$$

Defining[8]

$$M \equiv 2i\beta\partial Z\,, \tag{2.43}$$

with

$$Z(z)Z(w) \sim -\log(z-w)\,, \tag{2.44}$$

we have

$$\Sigma^{\pm} = e^{\pm i\beta Z}\Pi^{\pm}\,, \tag{2.45}$$

with

$$Z(z)\Pi^{\pm}(w) \sim 0\,. \tag{2.46}$$

But since $\Delta(\Sigma^{\pm}) = \beta^2/2$, it follows that $\Delta(\Pi^{\pm}) = 0$, hence $\Pi^{\pm} = 1$. Summing up, the spin fields have the form

$$S_r^{\pm} = e^{-ir(H_1\mp H_2)}e^{\pm i\sqrt{\frac{1}{2k}}Y}e^{\pm i\beta Z}\,. \tag{2.47}$$

2.2 Worldsheet symmetries

Since the spacetime theory is supersymmetric, the supercharges $Q_r^{\pm}$ take physical states into physical states, hence they should commute with the BRST operator. This means that in the OPEs between the worldsheet supercurrent G and $S_r^{\pm}$ no $(z-w)^{-3/2}$ terms appear. The supercurrent is given by

$$G = G_{AdS_3} + G_{\mathrm{U}(1)} + G_{\mathcal{N}/\,\mathrm{U}(1)}\,, \tag{2.48}$$

where

$$\begin{aligned} G_{AdS_3} &= \frac{2}{k}(\psi^A j_A + \frac{2i}{k}\psi^1\psi^2\psi^3)\,,\\ G_{\mathrm{U}(1)} &= J^0\psi^0\,, \end{aligned} \tag{2.49}$$

and by definition

$$J^i G_{\mathcal{N}/\,\mathrm{U}(1)} \sim \psi^i G_{\mathcal{N}/\,\mathrm{U}(1)} \sim 0\,. \tag{2.50}$$

In the computation of the OPEs between G and $S_r^{\pm}$, eqs. (2.33) and (2.50) imply that singular terms come only from

$$(G_{AdS_3} + G_{\mathrm{U}(1)})(z)\, e^{-ir(H_1\mp H_2)}e^{\pm i\sqrt{\frac{1}{2k}}Y}(w)\,, \tag{2.51}$$

and

$$G_{\mathcal{N}/\,\mathrm{U}(1)}(z)\, e^{(\pm i\beta Z)}(w)\,. \tag{2.52}$$

In the OPEs (2.51) two $(z-w)^{-3/2}$ terms appear which cancel each other. For the computation it is convenient to express (see (2.16), (2.24))

$$\begin{aligned} \frac{4i}{k^2}\psi^1\psi^2\psi^3 &= \frac{i}{\sqrt{k}}\partial H_1(e^{-iH_2} - e^{iH_2})\,,\\ J^0\psi^0 &= \frac{1}{\sqrt{2}}J^0(e^{-iH_2} + e^{iH_2})\,. \end{aligned} \tag{2.53}$$

[8]The sign choice in (2.42) is arbitrary.

Now, given the U(1) current M in (2.43), every operator Φ in the theory can be decomposed into terms with definite M-charge q as

$$\Phi = \sum_q : e^{\frac{iq}{2\beta} Z} P_q(M) : \tilde{\Phi}_q \,, \tag{2.54}$$

where $P_q(M)$ is a polynomial in $M(z)$ and its derivatives, and $\tilde{\Phi}_q M \sim 0$.

The absence of $(z-w)^{-\frac{3}{2}}$ terms in (2.52) and dimensional analysis imply that the only terms allowed when expressing $G_{\mathcal{N}/\,\mathrm{U}(1)}$ as (2.54) are

$$\begin{aligned} G_{\mathcal{N}/\,\mathrm{U}(1)} &= G^{+}_{\mathcal{N}/\,\mathrm{U}(1)} + G^{-}_{\mathcal{N}/\,\mathrm{U}(1)} \,, \\ G^{\pm}_{\mathcal{N}/\,\mathrm{U}(1)} &= \tau^{\pm} e^{\pm \frac{i}{2\beta} Z} \,, \end{aligned} \tag{2.55}$$

with

$$\begin{aligned} M(z) G^{\pm}_{\mathcal{N}/\,\mathrm{U}(1)}(w) &\sim \pm \frac{G^{\pm}_{\mathcal{N}/\,\mathrm{U}(1)}(w)}{z-w} \,, \\ Z(z)\tau^{\pm}(w) &\sim 0 \,. \end{aligned} \tag{2.56}$$

From this it follows that $\mathcal{N}/\,\mathrm{U}(1)$ has $N = 2$ supersymmetry, with M being the U(1) R-current. The rest of the $N = 2$ commutators can be obtained as in [11], using Jacobi identities. Note from (2.34) and (2.41) that M has the correct normalization for the $N = 2$ algebra, namely, $M(z)M(w) \sim \frac{c_{\mathcal{N}/\,\mathrm{U}(1)}}{3}(z-w)^{-2}$.

3. Spacetime $N = 3$ supersymmetry

Let us consider now the case of spacetime $N = 3$ supersymmetry in the NS sector. The global subalgebra is [4]

$$\{Q^a_r, Q^b_s\} = 2\delta^{a,b}\,,\, L_{r+s} + i\epsilon^{ab}{}_c (r-s) T^c_{r+s} \,, \tag{3.1}$$

$$[L_m, L_n] = (m-n) L_{m+n} \,, \tag{3.2}$$

$$[T^a_0, T^b_0] = i\epsilon^{ab}{}_c T^c_0 \,, \tag{3.3}$$

$$[L_m, Q^a_r] = \left(\frac{1}{2}m - r\right) Q^a_{m+r} \,, \tag{3.4}$$

$$[T^a_0, Q^b_r] = i\epsilon^{ab}{}_c Q^c_r \,, \tag{3.5}$$

where $m, n = 0, \pm 1$, $a, b, c = 1, 2, 3$ and $r, s = \pm 1/2$, and all other (anti)commutators vanish. All these spacetime operators are again obtained by contour integrals of dimension-1 local fields on the worldsheet. The three operators $L_{0,\pm 1}$ are given again by (2.5) -(2.6).

The three operators T^a_0 are the zero modes of the SU(2) R-current of the $N = 3$ algebra in spacetime. Let $\chi^a + \theta K^a$ be the three dimension-$\frac{1}{2}$ worldsheet supercurrents corresponding to T^a_0; we have, up to picture-changing,

$$T^a_0 = \oint K^a = \oint e^{-\phi} \chi^a \,. \tag{3.6}$$

The supercurrents $\chi^a + \theta K^a$ form an affine SU(2) superalgebra in $\mathcal{N}$, which is uplifted to an affine SU(2) algebra in the dual boundary CFT, according to the construction of [1]. The worldsheet level k' of this SU(2) affine algebra is again fixed by looking at the higher modes of the spacetime SU(2) affine currents, which are given by

$$T_n^a = \oint K^a \gamma^n \,, \tag{3.7}$$

and which satisfy

$$[T_m^a, T_n^b] = i\epsilon^{ab}{}_c T^c_{m+n} + \frac{k_{st}}{2}\, m\, \delta_{a,b}\, \delta_{n+m,0}\,, \tag{3.8}$$

where the spacetime SU(2) level is $k_{st} = k'p$ [1], with p given by (2.12). Consistency of the $N = 3$ algebra (see [4, 19]) in spacetime implies $k_{st} = \frac{2}{3}c_{st}$, which is equivalent to $k'p = \frac{2}{3}6kp$, and hence $k' = 4k$. Note that the level-dependent normalization of the T_0^a is the same as that of the R_0 operator in the $N = 2$ case (see eq. (2.8)). We will use these facts below.

We shall also use the purely bosonic σ-model and purely fermionic contributions to the total currents:

$$\begin{aligned} k^a &= K^a - k^a_{su(2)_2}\,, \\ k^a_{su(2)_2} &= -\frac{i}{4k}\epsilon^a{}_{bc}\chi^b\chi^c\,. \end{aligned} \tag{3.9}$$

The currents k^a and $k^a_{su(2)_2}$ are two commuting affine SU(2) currents, at levels $4k-2$ and 2, respectively, and the bosonic k^a commute with the fermions χ^a.

3.1 Properties of the spacetime supercharges

Again, the supercharges are given by

$$Q_r^a = b_r^a \oint e^{-\frac{\phi}{2}} S_r^a\,, \qquad r = \pm\frac{1}{2}\,. \tag{3.10}$$

In the Ramond sector, the algebra of the fermionic zero modes is now

$$\begin{aligned} \{\psi_0^A, \psi_0^B\} &= \eta^{AB} k/2\,, \\ \{\chi_0^a, \chi_0^b\} &= \delta^{a,b} 2k\,, \\ \{\psi_0^A, \chi_0^a\} &= 0\,. \end{aligned} \tag{3.11}$$

For the $SO(6)_2$ affine algebra obtained by Wick rotating ψ^3, we choose the Cartan subalgebra given by

$$\partial H_1 = \frac{2}{k}\psi^1\psi^2\,, \qquad \partial H_2 = -\frac{i}{k}\chi^3\psi^3\,, \qquad \partial H_3 = \frac{1}{2k}\chi^1\chi^2\,, \tag{3.12}$$

with

$$H_I(z)H_J(w) \sim -\delta_{IJ}\log(z-w)\,, \qquad I, J = 1, 2, 3\,, \tag{3.13}$$

and $H^\dagger_{1,3} = H_{1,3}$, $H_2^\dagger = -H_2$.

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As in the $N=2$ case, there is a basis for the spin fields of the form

$$S_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\lambda} = e^{i\varepsilon_1 H_1 + i\varepsilon_2(H_2+\pi N_1)+i\varepsilon_3(H_3+\pi N_2+\pi N_1)}\Lambda_\lambda\,, \tag{3.14}$$

with

$$\psi^A(z)\Lambda_\lambda(w)\sim 0\,, \qquad \chi^a(z)\Lambda_\lambda(w)\sim 0\,, \tag{3.15}$$

and

$$N_{1,2} = i\oint \partial H_{1,2}(z)\,. \tag{3.16}$$

The fields (3.14) provide a representation of (3.11) (as in (2.15)), with any other representation given by a linear change of basis of the spin fields. For future reference we remind that

$$\begin{aligned} e^{\pm iH_1} &= \frac{1}{\sqrt{k}}(\psi^1 \mp i\psi^2)\,, \\ e^{\pm i(H_2+\pi N_1)} &= \frac{1}{2\sqrt{k}}\chi^3 \mp \frac{1}{\sqrt{k}}\psi^3\,, \\ e^{\pm i(H_3+\pi N_2+\pi N_1)} &= \frac{1}{2\sqrt{k}}(\chi^1 \mp i\chi^2)\,, \end{aligned} \tag{3.17}$$

and from (3.9), we have

$$\begin{aligned} k^3_{su(2)_2} &= -i\partial H_3\,, \\ k^\pm_{su(2)_2} &= k^1_{su(2)_2} \pm ik^2_{su(2)_2} = \pm(e^{-iH_2}+e^{+iH_2})\,e^{\mp iH_3}\,. \end{aligned} \tag{3.18}$$

Now, the algebra (3.1)–(3.5) has three $N=2$ subalgebras, given by, say,

$$\begin{aligned} Q^\pm_r &= \frac{1}{\sqrt{2}}(Q^1_r \pm iQ^2_r)\,, \\ R_0 &= T^3_0\,. \end{aligned} \tag{3.19}$$

The other two $N=2$ subalgebras are obtained by cyclic permutations. From the result of the previous section we know that for the representation (3.14), the spin fields corresponding to the supercharges $Q^\pm_r$ in (3.19) must be those obtained from identifying $\varepsilon_1=-r$ and $\varepsilon_2=\pm r$ in (3.14), and that

$$J^A(z)\Lambda_\lambda(w)\sim 0\,. \tag{3.20}$$

The operators $\tilde\Sigma^\pm$ of (2.26), having charge ±1 under T^3_0 and satisfying (2.27), must be obtained from $e^{i\varepsilon_3 H_3}\Lambda_\lambda$. Since $e^{i\varepsilon_3 H_3}$ has charge $-\varepsilon_3=\pm1/2$ under T^3_0, the other $\pm1/2$ charge must be carried by[9] Λ_λ. We conclude that there are two fields $\Lambda^\pm$, which from (3.15) are charged under k^3 (see (3.9)) as

$$k^3(z)\Lambda^\pm(w)\sim \frac{\pm\frac{1}{2}\Lambda^\pm(w)}{z-w}\,, \tag{3.21}$$

[9]The possibility that $e^{i\varepsilon_3 H_3}$ carries charge $\mp1/2$ and Λ_λ carries charge $\pm3/2$ must be discarded because it is inconsistent with the action of $T^\pm_0$.

and that we should make the following identification (we omit the cocycles from now on):

$$\tilde{\Sigma}^{\pm} = e^{\mp \frac{i}{2} H_3} \Lambda^{\pm} . \tag{3.22}$$

Then from (2.27) we obtain

$$\begin{aligned} \Lambda^{+}(z)\Lambda^{-}(w) &\sim \frac{1}{(z-w)^{\frac{1}{2}}} , \\ \Lambda^{\pm}(z)\Lambda^{\pm}(w) &\sim O(w)(z-w)^{\frac{1}{2}} . \end{aligned} \tag{3.23}$$

A similar analysis to the one following (2.35), shows that in the first OPE (3.23) there exists a dimension-1 field $i\partial X^3$:

$$\Lambda^{+}(z)\Lambda^{-}(w) \sim \frac{1}{(z-w)^{\frac{1}{2}}} \left[1 - (z-w) \frac{i\partial X^3(w)}{\sqrt{2}} \right] , \tag{3.24}$$

with

$$X^3(z)X^3(w) \sim -\log(z-w) , \tag{3.25}$$

such that

$$\Lambda^{\pm} = e^{\mp \frac{i}{\sqrt{2}} X^3} . \tag{3.26}$$

From (3.21) and (3.23) we find the three-point function

$$\langle k^3(z_1)\Lambda^{+}(z_2)\Lambda^{-}(z_3)\rangle = \frac{1}{2} z_{12}^{-1}\, z_{13}^{-1}\, z_{23}^{1/2} , \tag{3.27}$$

and taking $z_{23} \to 0$, from (3.24) we have

$$k^3(z) i\partial X^3(w) \sim \frac{-1}{\sqrt{2}\,(z-w)^2} . \tag{3.28}$$

The above expression allows to write K^a as (see (3.9))

$$\begin{aligned} K^3 &= \hat{k}^3 + k^3_{su(2)_1} + k^3_{su(2)_2} , \\ K^{\pm} &= K^1 \pm iK^2 = \hat{k}^{\pm} + k^{\pm}_{su(2)_1} + k^{\pm}_{su(2)_2} , \end{aligned} \tag{3.29}$$

where

$$k^3_{su(2)_1} \equiv -\frac{i}{\sqrt{2}} \partial X^3 , \qquad k^{\pm}_{su(2)_1} \equiv k^1_{su(2)_1} \pm i k^2_{su(2)_1} = e^{\mp i\sqrt{2} X^3} . \tag{3.30}$$

Note that the $SU(2)_{4k}$ currents K^a are decomposed into three affine SU(2) currents $\hat{k}^a$, $k^a_{su(2)_1}$ and $k^a_{su(2)_2}$, with levels $4k-3$, 1 and 2, respectively. The $SU(2)_1$ is the theory of a single compact scalar X^3, and the $SU(2)_2$ is the theory of three free fermions whose currents are given in (3.9), (3.18). The three sets of SU(2) currents commute among themselves because

$$\hat{k}^a(z)\chi^b(w) \sim \hat{k}^a(z)\partial X^3(w) \sim \partial X^3(z)\chi^a(w) \sim 0 . \tag{3.31}$$

Summing up, from (3.22) and (3.26) we find that the spin fields corresponding to the supercharges in (3.19) are

$$S^{\pm}_r = e^{-ir(H_1 \mp H_2) \mp \frac{i}{2} H_3} e^{\mp \frac{i}{\sqrt{2}} X^3} , \tag{3.32}$$

and using the decomposition (3.29), the operators S_r^3 can now be obtained by applying $K^\pm$ to $S_r^\pm$. Defining

$$S_{[\pm,\pm,\pm,\pm]} = e^{\frac{i}{2}(\pm H_1 \pm H_2 \pm H_3) \pm \frac{i}{\sqrt{2}} X^3}\,, \tag{3.33}$$

the whole algebra is then generated by the following spin fields:

$$\begin{aligned}
S^-_{\frac{1}{2}} &= S_{[-,-,+,+]}\,,\\
S^3_{\frac{1}{2}} &= \frac{1}{2}\left(S_{[-,-,+,-]} + iS_{[-,+,-,+]}\right),\\
S^+_{\frac{1}{2}} &= -iS_{[-,+,-,-]}\,,\\
S^-_{-\frac{1}{2}} &= S_{[+,+,+,+]}\,,\\
S^3_{-\frac{1}{2}} &= \frac{1}{2}\left(S_{[+,+,+,-]} + iS_{[+,-,-,+]}\right),\\
S^+_{-\frac{1}{2}} &= -iS_{[+,-,-,-]}\,.
\end{aligned} \tag{3.34}$$

Note that the $e^{\frac{i}{2}(\pm H_1 \pm H_2 \pm H_3)}$ fields provide (several) spin-$\mathbf{\frac{1}{2}}$ representations of the $\mathrm{SU}(2)_2$ algebra of the χ^a. The fields $\Lambda^\pm$ in (3.26) are in turn two primaries of a spin-$\mathbf{\frac{1}{2}}$ representation of the $\mathrm{SU}(2)_1$ made out of X^3. The spin-$\mathbf{1}$ representation S_r^a emerges then as the symmetric part of the $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}}$ product of these two spin-$\mathbf{\frac{1}{2}}$ representations.

3.2 Worldsheet symmetries

Let us explore now the consequences induced by the structure of the spin fields on the properties of the worldsheet theory. The BRST condition again implies that no $(z-w)^{-\frac{3}{2}}$ singularities appear in the OPEs between G and the spin fields (3.34). The worldsheet supercurrent G is

$$G = G_{AdS_3} + G_{\mathrm{SU}(2)} + G_{\mathcal{N}/\,\mathrm{SU}(2)}\,, \tag{3.35}$$

with G_{AdS_3} given by (2.49) and

$$G_{\mathrm{SU}(2)} = \frac{1}{2k}\left(\chi^a k_a - \frac{i}{2k}\chi^1\chi^2\chi^3\right). \tag{3.36}$$

The terms whose OPE with the spin fields might give a $(z-w)^{-\frac{3}{2}}$ singularity can be written as

$$\begin{aligned}
G = \cdots &+ \frac{i}{\sqrt{k}}\partial H_1\left(e^{-iH_2} - e^{+iH_2}\right) + \frac{1}{2\sqrt{k}}\left(e^{-i\sqrt{2}X^3 + iH_3} + e^{i\sqrt{2}X^3 - iH_3}\right) - \\
&- \frac{1}{2\sqrt{k}}\left(\frac{i}{\sqrt{2}}\partial X^3 + i\partial H_3\right)\left(e^{-iH_2} + e^{+iH_2}\right) + G_{\mathcal{N}/\,\mathrm{SU}(2)}\,,
\end{aligned} \tag{3.37}$$

where the "dots" stand for terms that manifestly do not contribute $(z-w)^{-3/2}$ singularities. It can be checked that all the $(z-w)^{-3/2}$ terms in the OPEs cancel among themselves for the first three terms in (3.37). We find then that in the OPE between $e^{\mp\frac{i}{\sqrt{2}}X^3}$ and $G_{\mathcal{N}/\,\mathrm{SU}(2)}$ the highest singularity must be[10] $(z-w)^{-1/2}$. This result will be used in the next subsection.

[10]Note that from this fact we cannot deduce, as in section 2.2, the existence of an $N=2$ structure in $\mathcal{N}/\,\mathrm{SU}(2)$, with R-current $i\sqrt{2}\partial X^3$.

The identification (3.22), which can be stated more explicitly as

$$e^{\pm i\sqrt{\frac{1}{2k}}Y \pm i\beta Z} = e^{\mp\frac{i}{2}H_3 \mp \frac{i}{\sqrt{2}}X^3} \,, \tag{3.38}$$

has some consequences. From (2.8), (2.28), (3.6) and (3.19) we have

$$K^3 = i\sqrt{2k}\partial Y \,. \tag{3.39}$$

Let M^3 be the U(1) R-current of the worldsheet $N = 2$ structure obtained in the quotient $\mathcal{N}/\,\mathrm{U}(1)$, where the U(1) is generated by the supercurrent $\chi^3 + \theta K^3$. We have

$$M^3 = 2i\beta\partial Z \,, \tag{3.40}$$

where β and Z are defined in (2.34), (2.43), (2.44). Equating the exponents in (3.38) and using (3.18) and (3.30) it follows that

$$M^3 = -\frac{1}{k}K^3 + k^3_{su(2)_2} + 2k^3_{su(2)_1} \,. \tag{3.41}$$

It was shown in [8] that if $\mathcal{N}$ contains a supersymmetric $\mathrm{SU}(2)_{4k}$ affine symmetry and has the properties required to yield spacetime $N = 2$ supersymmetry, then a *sufficient* condition to extend spacetime supersymmetry to $N = 3$ is that $\mathcal{N}$ contains a dimension-1 field $k^3_{su(2)_1} = -\frac{i}{\sqrt{2}}\partial X^3$, which commutes with χ^a, and such that (3.41) holds. Here we see that the existence of the field X^3 with these properties is also *necessary.*

Moreover, eq. (3.41) was obtained by looking at the $N = 2$ subalgebra (3.19), but it must also hold for the other two $N = 2$ subalgebras of (3.1)–(3.5). The general relation is then

$$M^a = -\frac{1}{k}K^a + k^a_{su(2)_2} + 2k^a_{su(2)_1} \,, \tag{3.42}$$

where M^a is the U(1) R-current of the $N = 2$ structure obtained in the quotient $\mathcal{N}/\,\mathrm{U}(1)$ (with the U(1) generated by $\chi^a + \theta K^a$), and $k^a_{su(2)_{1,2}}$ were defined in (3.9) and (3.30). Of course, for $a = 1, 2$, in order to obtain (3.42) explicitly in the same way (3.41) was obtained from (3.38), different basis for the spin fields must be chosen.

An important consequence of (3.42) is that

$$\begin{aligned} K^a(z)M^b(w) &\sim \frac{i\epsilon^{ab}{}_c M^c(w)}{z-w} \,, \\ M^d(z)\chi^e(w) &\sim \frac{(1-\frac{1}{k})\; i\epsilon^{de}{}_f \chi^f}{z-w} \,. \end{aligned} \tag{3.43}$$

Summing up, we have seen that the existence of $N = 3$ superconformal symmetry in the two dimensional theory dual to $AdS_3 \times \mathcal{N}$ implies that:

1. $\mathcal{N}$ contains an affine SU(2) symmetry at level $4k$.

2. In the quotient of $\mathcal{N}$ by each one of the three SU(2) supercurrents $\chi^a + \theta K^a$, the worldsheet supersymmetry is extended to $N = 2$, and the corresponding three U(1) R-currents M^a satisfy (3.43).

Moreover, it is easy to see that these two conditions are also *sufficient* to yield $N = 3$ spacetime supersymmetry, and are equivalent to the conditions formulated in [8].

Note that the fact that the SU(2) currents K^a rotate the three U(1) R-currents M^a implies that the SU(2) part is embedded in $\mathcal{N}$ as a nontrivial fibration over $\mathcal{N}/\,\mathrm{SU}(2)$. Namely, if $\mathcal{N}$ is a direct product — $\mathcal{N} = \mathrm{SU}(2) \times \frac{\mathcal{N}}{\mathrm{SU}(2)}$ — then K^a commute with the contributions to M^a coming from $\frac{\mathcal{N}}{\mathrm{SU}(2)}$, and thus cannot satisfy (3.43). The only exception to this is when $\frac{\mathcal{N}}{\mathrm{SU}(2)}$ is trivial, namely, $\mathcal{N} = \mathrm{SU}(2)$. This holds for $k = 3/4$ ($k' = 3$), and the conditions (3.43) are indeed satisfied in this case. This is a special case of a series of solutions $\mathcal{N} = \mathrm{SU}(3)_{k'}/\,\mathrm{U}(1)$ [8], because $\mathrm{SU}(2)_3 \simeq \mathrm{SU}(3)_3/\,\mathrm{U}(1)$ as SCFTs.

3.3 Spacetime $N = 1$ supersymmetry

An interesting consequence of the structure found above is that in every background which allows spacetime $N = 3$ supersymmetry, there is also a different GSO projection leading to precisely $N = 1$ boundary supersymmetry.[11] This algebra is constructed from the singlet of the $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}}$ (discussed after eq. (3.34)), and is generated by

$$\begin{aligned} S_{\frac{1}{2}} &= \frac{1}{2}\left(S_{[-,+,+,-]} - iS_{[-,-,-,+]}\right), \\ S_{-\frac{1}{2}} &= \frac{1}{2}\left(S_{[+,-,+,-]} - iS_{[+,+,-,+]}\right). \end{aligned} \tag{3.44}$$

Note that the difference between these two operators and $S^3_{\pm\frac{1}{2}}$ in (3.34) is that the sign of H_2 is changed (different GSO projection) and the relative sign between the terms is now negative for the singlet. Using the fact that in the OPE between $e^{\mp\frac{i}{\sqrt{2}}X^3}$ and $G_{\mathcal{N}/\,\mathrm{SU}(2)}$, the highest singularity is $(z-w)^{-\frac{1}{2}}$ (see the discussion after eq. (3.37)), the operators (3.44) can be checked to be BRST invariant. Finally, they have regular OPEs with K^a, as expected for the singlet.

4. Spacetime small and large $N = 4$ supersymmetry

The analysis for these two cases is similar, mutatis mutandi, to that of $N = 2, 3$, and we will only indicate the general arguments.

Small $N = 4$ has an affine SU(2) R-symmetry whose level is fixed to be k (the same as the AdS_3 level) by the spacetime algebra. The central charge of $\mathcal{N}/\,\mathrm{SU}(2)_k$ is then

$$c_{\mathcal{N}/\,\mathrm{SU}(2)_k} = c_{\mathcal{N}} - \frac{3(k-2)}{k} - \frac{3}{2} = 6\,, \tag{4.1}$$

and the construction of the spin fields leads to the same supercharges constructed in [1]. As in section 2, the BRST condition shows that the worldsheet supersymmetry in $\mathcal{N}/\,\mathrm{SU}(2)_k$ is extended to $N = 2$, and from (4.1) this actually means that there is a small $N = 4$ supersymmetry in the worldsheet.[12] The latter was realized in [1] by means of $\mathcal{N}/\,\mathrm{SU}(2)_k = T^4$.

The analysis of the large $N = 4$ case leads necessarily to $\mathcal{N} = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ [10]. The reason is that the spacetime R-symmetry requires the presence of affine

[11] Indeed, this was shown to be the case for the examples studied in [9, 8]; here we argue that this is general.

[12] A $c = 6$, $N = 2$ SCFT has necessarily small $N = 4$ [12].

Boundary	R-current	Conditions on $\mathcal{N}$
$N=2$	$\mathrm{U}(1)$	$\mathcal{N} \supset \mathrm{U}(1)$ and $\mathcal{N}/\,\mathrm{U}(1)$ has $N=2$ SUSY.
$N=3$ $(N=1)$	$\mathrm{SU}(2)$	$\mathcal{N} \supset \mathrm{SU}(2)_{4k}$ and each $\mathcal{N}/K^a$ has $N=2$ SUSY with $\mathrm{U}(1)$ R-currents M^a satisfying (3.43).
Small $N=4$	$\mathrm{SU}(2)$	$\mathcal{N} = \mathrm{SU}(2)_k \times M^4$
Big $N=4$	$\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$	$\mathcal{N} = \mathrm{SU}(2)_{k'}\times\mathrm{SU}(2)_{k''}\times\mathrm{U}(1),\quad 1/k = 1/k' + 1/k''$

Table 1: This summarizes the necessary conditions on the internal CFT $\mathcal{N}$ imposed by different extended supersymmetries in the dual boundary CFT. All the conditions are also sufficient. In the $N=3$ case, $N=1$ is obtained for the other GSO projection; K^a are the generators of $\mathrm{SU}(2)_{4k}$, and $\mathcal{N}/K^a$ denotes the $\mathcal{N}/\,\mathrm{U}(1)$ quotient SCFT with the $\mathrm{U}(1)$ generated by the superfield $\chi^a + \theta K^a$. In small $N=4$, M^4 is a $c=6$ unitary CFT with a small $N=4$ superconformal symmetry.

$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ in the worldsheet theory, and consistency of the spacetime algebra requires the levels k', k'' of the two $\mathrm{SU}(2)$ models to satisfy $1/k = 1/k' + 1/k''$. This implies that the $(AdS_3)_k \times \mathrm{SU}(2)_{k'} \times \mathrm{SU}(2)_{k''} \times \mathrm{U}(1)$ background is critical, and thus the unique one allowing large $N=4$ boundary supersymmetry. This is compatible with the fact that the spin fields generating the Ramond sector of the 10 free worldsheet fermions already have $\Delta = 5/8$, and hence the spacetime supercurrents must be constructed from them. This is indeed the construction in [10]. This model was also studied in [25, 26].

5. Discussion

In this work the necessary conditions have been found for the internal CFT $\mathcal{N}$, imposed by the existence of $N=2,3,4$ SCFT in the boundary dual of string theory on $AdS_3 \times \mathcal{N}$. Our results are summarized in table 1.

The case of *precisely* $N=1$ supersymmetry in the boundary two dimensional CFT seems to be non-trivial. Of course, it can be realized as a byproduct of theories having higher amounts of boundary supersymmetries. Examples of this are the other GSO projection of theories with boundary $N=3$ (see section 3.3), or a particular Z_2 orbifold of theories with $N=2$ [6]. Recently, the work in ref. [27] implies that if the SCFT $\mathcal{N}$ contains a tri-critical Ising model, then $N=1$ supersymmetry can be constructed in the boundary dual of AdS_3. A new family of examples of this sort is $\mathcal{N} = \mathrm{SO}(7)_{k'}/(G_2)_{k'+1}$, which leads to precisely $N=1$ two dimensional supersymmetry in spacetime.[13]

Finally, general properties of the spectrum of superstrings in $AdS_3 \times \mathcal{N}$, with extended supersymmetry, can be studied along the lines of [28, 29].

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A. Properties of the affine currents

In this appendix we shall prove three properties of the affine currents which we have used throughout the paper. Let $\chi + \theta K$ be a generic worldsheet supercurrent corresponding to the affine symmetries uplifted to the boundary theory, then it holds for χ and K that:

1. They are orthogonal.
2. They have the same normalization.
3. They commute with the three $\mathrm{SL}(2,\mathbb{R})$ currents, so they lie entirely in $\mathcal{N}$.

Given the worldsheet supercurrent G, remember that for any dimension-$\frac{1}{2}$ superfield $\chi+\theta K$, we have

$$\begin{aligned} [G_s, K_m] &= -m\chi_{s+m}\,, \\ \{G_s, \chi_n\} &= K_{s+n}\,. \end{aligned} \tag{A.1}$$

Property 1 can be seen by considering the Jacobi identity

$$\begin{aligned} \{G_s,[\chi_t,K_q]\} - \{\chi_t,[K_q,G_s]\} + [K_q,\{G_s,\chi_t\}] &= 0\,, \\ \{G_s,[\chi_t,K_q]\} - q\{\chi_t,\chi_{q+s}\} + [K_q,K_{s+t}] &= 0\,, \\ \{G_s,[\chi_t,K_q]\} - q\delta_{t+q+s,0} + q\delta_{t+q+s,0} &= 0\,, \end{aligned} \tag{A.2}$$

for every set of s,t,q, and it follows that

$$K(z)\chi(w) \sim 0\,. \tag{A.3}$$

Property 2 follows from

$$\begin{aligned} \{G_{-m-n},[\chi_n,K_m]\} - \{\chi_n,[K_m,G_{-m-n}]\} + [K_m,\{G_{-m-n},\chi_n\}] &= 0\,, \\ 0 - m\{\chi_n,\chi_{-n}\} + [K_m,K_{-m}] &= 0\,, \end{aligned} \tag{A.4}$$

for every m,n.

As for property 3, the commutators $[L_{0,\pm 1}, R_0] = 0$, where $L_{0,\pm 1}$ are given in (2.5) and $R_0 \equiv \oint K$, show that there are no simple poles in the OPEs between J^A and K. A possible double pole is forbidden by Jacobi identities such as

$$\begin{aligned} [K_n,[J^A_{-n+1},J^B_{-1}]] + [J^A_{-n+1},[J^B_{-1},K_n]] + [J^B_{-1},[K_n,J^A_{-n+1}]] &= 0\,, \\ \varepsilon^{ABC}\eta_{CD}[K_n,J^D_{-n}] + 0 + 0 &= 0\,, \end{aligned} \tag{A.5}$$

and from this we find that

$$J^A(z)K(w) \sim 0\,. \tag{A.6}$$

Consider now the Jacobi identity,

$$\begin{aligned}
[G_s,[J^A_m,K_n]] + [J^A_m,[K_n,G_s]] + [K_n,[G_s,J^A_m]] &= 0\,,\\
0 + n[J^A_m,\chi_{n+s}] - m[K_n,\psi^A_{m+s}] &= 0\,,
\end{aligned} \tag{A.7}$$

where the first term is zero because of (A.6). Choosing $n = 0$ or $m = 0$ in (A.7), we have then

$$[K_0,\psi^A_t] = [J^A_0,\chi_t] = 0\,, \tag{A.8}$$

for every t. On the other hand, in the OPE, say, between K and ψ^A, the only terms that can appear are

$$K(z)\psi^A(w) \sim \frac{\eta(w)}{z-w}\,, \tag{A.9}$$

with $\Delta(\eta) = 1/2$, but from (A.8),

$$\eta(w) = [K_0,\psi^A(w)] = 0\,. \tag{A.10}$$

In the same way we see that

$$\begin{aligned}
J^A(z)\chi(w) &\sim 0\,,\\
\psi^A(z)K(w) &\sim 0\,.
\end{aligned} \tag{A.11}$$

Consider finally Jacobi identities such as

$$\begin{aligned}
\{G_{-n+1},[\chi_n,J^A_{-1}]\} - \{\chi_n,[J^A_{-1},G_{-n+1}]\} + [J^A_{-1},\{G_{-n+1},\chi_n\}] &= 0\,,\\
0 + \{\chi_n,\psi^A_{-n}\} + [J^A_{-1},K_1] &= 0\,,\\
0 + \{\chi_n,\psi^A_{-n}\} + 0 &= 0\,,
\end{aligned} \tag{A.12}$$

where we have used (A.6) and (A.11). We conclude then that

$$\psi^A(z)\chi(w) \sim 0\,, \tag{A.13}$$

and it follows from (A.6), (A.11) and (A.13) that $\chi + \theta K$ commutes with the $\mathrm{SL}(2,\mathbb{R})$ sector of the worldsheet, and hence it lies entirely in $\mathcal{N}$.

B. Realization of the symmetries for $N = 3$

In [8] it was shown that eq. (3.41) holds in case $\mathcal{N} = \mathrm{SU}(3)/\,\mathrm{U}(1)$ and $\mathcal{N} = \mathrm{SO}(5)/\,\mathrm{SO}(3)$. These backgrounds, as shown there explicitly, yield spacetime $N = 3$ (or $N = 1$) supersymmetry.

In this appendix we will illustrate our results by seeing how the geometric structure obtained in section 3.2 is explicitly realized in another background which was also shown to have $N = 3$ spacetime supersymmetry.

The background is $\mathcal{N} = (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1))/Z_2$, and was studied in [9], where the explicit form of the spin fields is given. The two supersymmetric SU(2) WZW models have level $2k$. Let $\chi^a_{1,2} + \theta K^a_{1,2}$ be the SU(2) currents and $\lambda + \theta i\partial Y$ be the affine U(1) current. The Z_2 orbifold acts as

$$\begin{aligned} (K^a_1, K^a_2, Y) &\longrightarrow (K^a_2, K^a_1, -Y)\,, \\ (\chi^a_1, \chi^a_2, \lambda) &\longrightarrow (\chi^a_2, \chi^a_1, -\lambda)\,. \end{aligned} \tag{B.1}$$

Define

$$\begin{aligned} K^a_+ &= K^a_1 + K^a_2 \qquad & \chi^a_+ &= \chi^a_1 + \chi^a_2\,, \\ K^a_- &= K^a_1 - K^a_2 \qquad & \chi^a_- &= \chi^a_1 - \chi^a_2\,. \end{aligned} \tag{B.2}$$

The currents $\chi^a_+ + \theta K^a_+$ form a supersymmetric SU(2) WZW model at level $4k$ which survives the orbifold and is uplifted to the SU(2) affine R-current of the $N = 3$ superconformal algebra in spacetime.

The supercurrent is

$$G_{\mathcal{N}} = \frac{1}{k}\left(\chi^a_1 k_{1a} - \frac{i}{k}\chi^1_1\chi^2_1\chi^3_1\right) + \frac{1}{k}\left(\chi^a_2 k_{2a} - \frac{i}{k}\chi^1_2\chi^2_2\chi^3_2\right) + i\lambda\partial Y\,. \tag{B.3}$$

Each superfield $\chi^a_+ + \theta K^a_+$ constitutes a supersymmetric U(1) affine model with supercurrent

$$G_{U^a(1)} = \frac{1}{2k}\chi^a_+ K^a_+\,. \tag{B.4}$$

The coset of $\mathcal{N}$ by each of these U(1) currents gives a theory whose supercurrent can be written as

$$\begin{aligned} G_{\mathcal{N}/U^3(1)} = G_{\mathcal{N}} - G_{U^3(1)} = & \frac{1}{2k}\chi^1_+ k^1_+ + \frac{1}{2k}\chi^2_+ k^2_+ + \\ & + \frac{1}{2k}\chi^1_- k^1_- + \frac{1}{2k}\chi^2_- k^2_- + \frac{1}{2k}\chi^3_- K^3_- + i\lambda\partial Y\,, \end{aligned} \tag{B.5}$$

with the $a = 1, 2$ cases given by cyclic permutations. For each a there is an $N = 2$ structure which survives the orbifold, whose U(1) R-current is

$$M^a = -\frac{1}{k}K^a_+ - \frac{i}{4k}\epsilon^a{}_{bc}\chi^b_+\chi^c_+ - \frac{i}{4k}\epsilon^a{}_{bc}\chi^b_-\chi^c_- - \frac{i}{\sqrt{2k}}\chi^a_-\lambda\,, \tag{B.6}$$

and from (3.42) we can identify

$$2k^a_{su(2)_1} = -\frac{i}{4k}\epsilon^a{}_{bc}\chi^b_-\chi^c_- - \frac{i}{\sqrt{2k}}\chi^a_-\lambda\,. \tag{B.7}$$

The currents (B.6) clearly satisfy

$$\begin{aligned} K^a_+(z)M^b(w) &\sim \frac{i\epsilon^{ab}{}_c M^c(w)}{z-w}\,, \\ M^d(z)\chi^e_+(w) &\sim \frac{\left(1-\frac{1}{k}\right) i\epsilon^{de}{}_f\chi^f_+}{z-w}\,, \end{aligned} \tag{B.8}$$

as expected from (3.43).

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